Autonomous Dynamical Systems, Explicitly Time Dependent First Integrals, and Gradient Systems

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We study first integrals and first-order autonomous systems of differential equations. Some of the concepts for explicitly time-independent first integrals are extended to explicitly time-dependent first integrals. Several applications are given.

Key words: Dynamical System; Time Dependent First Integrals; Gradient Systems.

We investigate first-order autonomous systems of differential equations

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = f_j(\mathbf{u}), \qquad j = 1,2,\ldots,n, \tag{1}$$

and first integrals in particular explicitly time-dependent first integrals [1]. It is assumed that the functions $f_j : \mathbb{R}^n \to \mathbb{R}$ are analytic. If the autonomous system (1) admits a first integral I, it can be written as [2]

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = S(\mathbf{u})\nabla I(\mathbf{u}), \tag{2}$$

where $\nabla I(\mathbf{u}) := (\partial I/\partial u_1,\ldots,\partial I/\partial u_n)^{\mathrm{T}}$, $S^{\mathrm{T}}(\mathbf{u}) = -S(\mathbf{u})$ is a skew-symmetric $n\times n$ matrix and $^{\mathrm{T}}$ denotes transpose. This representation is important when we discretize a dynamical system (1) and want to preserve the first integral. As an example consider

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = cu_1 + c_{23}u_2u_3, \tag{3a}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = cu_2 + c_{31}u_3u_1, \tag{3b}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = cu_3 + c_{12}u_1u_2, \tag{3c}$$

where $c_{23}, c_{31}, c_{12} \neq 0$, and c describes the damping. If $c = 0$, then the system admits the first integral $I_1 = (c_{31}u_1^2 - c_{23}u_2^2)/2$ and the dynamical system can be written as

$$\begin{pmatrix} \mathrm{d}u_1/\mathrm{d}t \\ \mathrm{d}u_2/\mathrm{d}t \\ \mathrm{d}u_3/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} 0 & -u_3 & 0 \\ u_3 & 0 & c_{12}u_1/c_{23} \\ 0 & -c_{12}u_1/c_{23} & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial I_1/\partial u_1 \\ \partial I_1/\partial u_2 \\ \partial I_1/\partial u_3 \end{pmatrix}. \tag{4}$$

Note that the system also admits the independent first integral $(c_{12}u_1^2 - c_{23}u_3^2)/2$, and the dynamical system can be reconstructed using Nambu mechanics. If $c \neq 0$, then I_1, I_2 are not first integrals anymore.

Now many dissipative dynamical systems such as the Lorenz model and the Rikitake-two disc dynamo admit, depending on the bifurcation parameters, explicitly time-dependent first integrals. These type of first integrals are of the form [3 – 9]

$$f(\mathbf{u}(t))\mathrm{e}^{\lambda t}. \tag{5}$$

The dynamical system given above with $c \neq 0$ admits first integrals of this form. Thus the concepts given above to cast the dynamical system (1) into the form (2) has to be extended. We show with an example that the concept to write system (1) if a first integral exists as a gradient system (2) can be extended to explicitly time-dependent first integrals of the type given by (5). To study such cases we extend the autonomous system (1) to

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = \mathbf{f}(\mathbf{u}), \quad \frac{\mathrm{d}u_{n+1}}{\mathrm{d}\tau} = 1. \tag{6}$$

Thus we set $t \to u_{n+1}$ and $u_n(\tau = 0) = 0$. The first integrals (5) then take the form $f(u_1,\ldots,u_n)\mathrm{e}^{\lambda u_{n+1}}$.

As an example consider the autonomous system

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_1(1 + au_2 + bu_3), \tag{7a}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = u_2(1 - au_1 + cu_3), \tag{7b}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = u_3(1 - bu_1 - cu_2), \tag{7c}$$

where $a,b,c \in \mathbb{R}$ with the first integral $I(t,\mathbf{u}(t)) = (u_1+u_2+u_3)\mathrm{e}^{-t}$. Thus we consider the autonomous system

$$\frac{\mathrm{d}u_1}{\mathrm{d}\tau} = u_1(1+au_2+bu_3), \tag{8a}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}\tau} = u_2(1-au_1+cu_3), \tag{8b}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}\tau} = u_3(1-bu_1-cu_2), \tag{8c}$$

$$\frac{\mathrm{d}u_4}{\mathrm{d}\tau} = 1. \tag{8d}$$

This system can be written in the form

$$\begin{pmatrix} \mathrm{d}u_1/\mathrm{d}\tau \\ \mathrm{d}u_2/\mathrm{d}\tau \\ \mathrm{d}u_3/\mathrm{d}\tau \\ \mathrm{d}u_4/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} 0 & s_{12}\mathrm{e}^{u_4} & s_{13}\mathrm{e}^{u_4} & s_{14}\mathrm{e}^{u_4} \\ -s_{12}\mathrm{e}^{u_4} & 0 & s_{23}\mathrm{e}^{u_4} & s_{24}\mathrm{e}^{u_4} \\ -s_{13}\mathrm{e}^{u_4} & -s_{23}\mathrm{e}^{u_4} & 0 & s_{34}\mathrm{e}^{u_4} \\ -s_{14}\mathrm{e}^{u_4} & -s_{24}\mathrm{e}^{u_4} & -s_{34}\mathrm{e}^{u_4} & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial I/\partial u_1 \\ \partial I/\partial u_2 \\ \partial I/\partial u_3 \\ \partial I/\partial u_4 \end{pmatrix} \tag{9}$$

with $I(\mathbf{u}) = (u_1+u_2+u_3)\mathrm{e}^{-u_4}$, $\partial I/\partial u_1 = \partial I/\partial u_2 = \partial I/\partial u_3 = \mathrm{e}^{-u_4}$, $\partial I/\partial u_4 = -(u_1+u_2+u_3)\mathrm{e}^{-u_4}$, and

$$s_{12} = \frac{1}{3}u_1 - \frac{1}{3}u_2 + au_1u_2, \tag{10a}$$

$$s_{13} = \frac{1}{3}u_1 - \frac{1}{3}u_3 + bu_1u_3, \tag{10b}$$

$$s_{23} = \frac{1}{3}u_2 - \frac{1}{3}u_3 + cu_2u_3, \tag{10c}$$

where $s_{14} = s_{24} = s_{34} = -1/3$. Thus from the example it is obvious how to extend the concept to explicitly time-dependent first integrals.

The approach can be extended if there are more than one explicitly first integral. Consider the dynamical system

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = cu_1 + c_{234}u_2u_3u_4, \tag{11a}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = cu_2 + c_{134}u_1u_3u_4, \tag{11b}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = cu_3 + c_{124}u_1u_2u_4, \tag{11c}$$

$$\frac{\mathrm{d}u_4}{\mathrm{d}t} = cu_4 + c_{123}u_1u_2u_3. \tag{11d}$$

This is an extension of system (3) to higher dimensions. This system has been studied for $c=0$ from a Lie algebraic and integrability point of view by Steeb [10]. Such a system for $c=0$ appears from the self-dual Yang–Mills equation by exact reduction [11]. For $c=0$, we have the three first integrals

$$c_{134}u_1^2 - c_{234}u_2^2,\ \ c_{124}u_2^2 - c_{134}u_3^2,\ \ c_{123}u_3^2 - c_{124}u_4^2. \tag{12}$$

Including damping $c \neq 0$ provides the explicitly time-dependent first integrals

$$\mathrm{e}^{-ct}(c_{134}u_1^2 - c_{234}u_2^2),\ \ \mathrm{e}^{-ct}(c_{124}u_2^2 - c_{134}u_3^2),\ \ \mathrm{e}^{-ct}(c_{123}u_3^2 - c_{124}u_4^2). \tag{13}$$

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